Journal of Modern Technology and Engineering
Vol.4, No.3, 2019, pp.153-159

INDEPENDENCE SATURATION IN SOME TREE NETWORKS

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Abstract. Let $G = (V, E)$ be a graph and $v \in V$. Let $IS(v)$ denote the maximum cardinality of an independent set in G which contains v. Then $IS(G) = \min\{IS(v) : v \in V\}$ is called *the independence saturation number* of G. In this paper, we examine independence saturation number of some trees such as E_p^t graph, double comet, double star, binomial tree and generalized caterpillar graphs.

Keywords: Graph theory, independent sets, independence saturation number, tree.
AMS Subject Classification: 68R10, 05C69, 05C05.
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Received: 9 August 2019; Accepted: 25 October 2019; Published: 25 December 2019.

1 Introduction

The maximum independent set problem, a fundamental subject of theoretical computer science and discrete mathematics, is not only theoretical but also has wide applications in a variety of fields. However, determining the size of a maximum independence set is NP-hard for a general graph. It is also even more difficult to calculate exactly the number of the all maximum independence set.

An independent set of a graph G with vertex set V is a subset I of V, such that each pair of vertices in I is not adjacent in G. The maximum independent set is an independent set. But it is not a subset of other independent sets. The independent set with the largest element is called the maximum independent set. The independence number of graph G is the cardinality of a maximum independent set. The solutions of many fundamental graph problems depend on the maximum independent set. For instance, the minimum vertex covering problem of a graph or the maximum clique problem on the complement of the graph is the maximum independent set problem for the same graph (Karp, 1972) and (Pardalos & Xue, 1994). Furthermore, the maximum independent set problem is also important for the graph coloring, calculating of the maximum common induced subgraphs, and finding the maximum common edge subgraphs (Liu et al., 2015). In addition to its theoretical importance, the maximum independent set problem has been used in important applications in various fields, such as coding theory (Butenko et al., 2002), collusion detection in voting pools (Araujo et al., 2011), scheduling in wireless network (Joo et al., 2016).

Since the maximum independent set problem has wide applications in various fields, many versions of it have been identified and studied in the literature to this day. In this article, the independence saturation number recently described by Subramanian is discussed (Arumugam & Subramanian, 2007). For a vertex v of a graph G, let $IS(v)$ denote the maximum cardinality of an independent set in G which contains v. Hence, the independence saturation number of G, $IS(G)$, is the minimum cardinality of a $IS(v)$ for every $v \in V$. That is, $IS(G) = \min\{IS(v) : v \in V\}$. Thus $IS(G)$ is the largest positive integer k such that every vertex of G lies in an independent set of cardinality k . Let $v \in V$ be such that $IS(v) = IS(G)$. Then any independent set of cardinality $IS(G)$ containing v is called an $IS - set$.

In this paper, we consider simple finite undirected graphs. Let $G = (V, E)$ be a graph with a vertex set $V = V(G)$ and an edge set $E = E(G)$. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them. If u and v are not connected, then $d(u, v) = \infty$, and for $u = v$, $d(u, v) = 0$. The *eccentricity* of a vertex v in G is the distance from v to a vertex farthest away from v in G. The *diameter* of G, denoted by $diam(G)$, is the largest distance between two vertices in V. The *degree* $\deg_G(v)$ of a vertex $v \in V$ is the number of edges incident to v. The *maximum degree* of G is $\Delta(G) = \max\{\deg_G(v) | v \in V\}$. The *minimum degree* of G is $\delta(G) = \min\{\deg_G(v) | v \in V\}$. For any vertex $v \in V$, *the open neighbourhood of* v is $N_G(v) = \{u \in V | uv \in E\}$ and *closed neighbourhood of* v is $N_G[v] = N_G(v) \cup \{v\}$. A vertex of degree zero is an *isolated vertex* or an *isolate*. A vertex of degree one is called a *leaf* or an *endvertex*, and its neighbor is called a *support vertex*. For any leaf vertex v and support vertex w, the edge vw is called a *pendant edge* (Chartrand & Lesniak, 2005; Buckley & Harary, 1990).

The paper proceeds as follows. In section 2, existing literature on independence saturation number is reviewed. In section 3, the independence saturation numbers for some tree graphs such as E_p^t graph, generalized caterpillar graph, double star graph, binomial tree and double comet graph are computed and exact formulae are derived.

2 Known Results

In this section, we give some of the known result on independence saturation number.

Theorem 1. *(Arumugam & Subramanian, 2007) If G is an $r-regular$ graph on n vertices with $r > 0$, then $IS(G) \leq n/2$. Further equality holds if and only if G is bipartite.*

Theorem 2. *(Arumugam & Subramanian, 2007) For any graph G on n vertices, $IS(G) \leq n - \Delta(G)$. Further for a tree T, $IS(G) \leq n - \Delta(G)$, if and only if $V - N_G(v)$ is an independent set for every vertex v of degree $\Delta(G)$ and $p_u \leq p_v$ for every $u \in N_G(v)$, where p_x is the number of pendant vertices adjacent to x.*

Theorem 3. *(Arumugam & Subramanian, 2007) The independence saturation of*

i. for the cycle graph C_n, $IS(C_n) = \lfloor n/2 \rfloor$.

ii. for the complete graph K_n, $IS(K_n) = 1$.

iii. for the complete bipartite graph $K_{m,n}$, $IS(K_{m,n}) = \min\{m, n\}$.

iv. for the star graph $K_{1,n}$, $IS(K_{1,n}) = 1$.

Theorem 4. *(Arumugam & Subramanian, 2007) Let G be any graph on n ($n \geq 3$) vertices. Then*

i. $3 \leq IS(G) + IS(\overline{G}) \leq n + 1 - (\Delta - \delta)$ and $2 \leq IS(G). IS(\overline{G}) \leq (n - \Delta)(\delta + 1)$.

ii. The following are equivalent.

(a) $IS(G) + IS(\overline{G}) = 3$.

(b) $IS(G). IS(\overline{G}) = 2$.

(c) G or $\overline{G}$ has the property that it has a unique vertex of degree $n - 1$ and has at least one pendant vertex.

iii. $IS(G) + IS(\overline{G}) = n + 1$ if and only if G is either K_n or $\overline{K_n}$.

Theorem 5. *(Berberler & Berberler, 2018) The independence saturation of* *(a) the path P_n $(n \geq 2)$ is $\lfloor n/2 \rfloor$,*

(b) the wheel W_n is 1,

(c) the comet $C_{t,r}$ is $\lceil t/2 \rceil$.

Corollary 1. *(Berberler & Berberler, 2017) If a vertex u has eccentricity one in graph G, then $IS(u) = 1$.*

Corollary 2. *(Berberler & Berberler, 2017) Let G be a graph with n vertices. If G has a vertex with eccentricity one, then $IS(G) = 1$.*

3 Independence Saturation in Some Trees

In this section, the independence saturation numbers for some tree graphs such as E_p^t graph, generalized caterpillar graph, double star graph, binomial tree and double comet graph are computed and exact formulae are derived.

Definition 1. *(Cormen et al., 1990) The graph E_p^t has t legs and each leg has p vertices (Figure 1). Thus E_p^t has $pt+2$ vertices and $pt+1$ edges.*

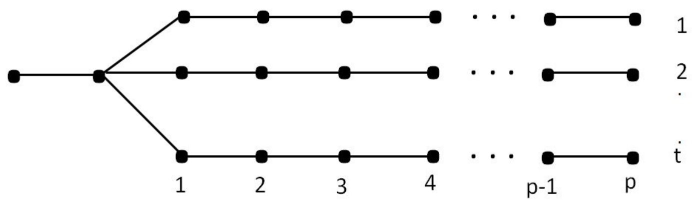

Figure 1: E_p^t graph with $pt+2$ vertices

Theorem 6. *Let G be a E_p^t graph. Then, $IS(G) = t\lfloor p/2 \rfloor$.*

Proof. Let label the vertices of E_p^t by x, y and v_{ij} where $i \in \{1, 2, \ldots, t\}$, $j \in \{1, 2, \ldots, p\}$, x is the vertex having maximum degree and y is the adjacent to x of degree 1. Let I be a $IS-$ set of E_p^t graph. Hence, we have three cases depending on the different vertex types of E_p^t.

Case 1. Let I be an $IS-$ set containing the vertex y of the graph E_p^t. Then I contains y and all vertices with odd subscripts of the path graph whose number is t. Therefore, $I = \{y\} \cup \left\{v_{i(2n-1)} | 1 \leq i \leq t, 1 \leq n \leq \lceil p/2 \rceil \right\}$, implying that $IS(y) = 1 + \lceil p/2 \rceil$.

Case 2. Let I be an $IS - set$ containing the vertex x of the graph E_p^t. Then I contains x and all vertices with even subscripts of the path graph whose number is t. Therefore, $I = \{x\} \cup \left\{v_{i(2n)} | \, 1 \leq i \leq t, \, 1 \leq n \leq \lfloor p/2 \rfloor \right\}$, implying that $IS(y) = 1 + \lfloor p/2 \rfloor$.

Case 3. Let I be an $IS - set$ containing the vertex v_{ij} of the graph E_p^t. For this, it is sufficient to consider the vertices in only one of the path graphs whose number is t. Hence, it is easily see that $IS(v_{ij}) = \lfloor p/2 \rfloor$ by Theorem 5(a). However, either the vertex x or y of the graph must be added to this set. Thus, we get $IS(v_{ij}) = 1 + t \lfloor p/2 \rfloor$.

By *Cases 1, 2,* and *3*, we have $IS(G) = \min \{1 + t \lfloor p/2 \rfloor, 1 + t \lceil p/2 \rceil\} = 1 + t \lfloor p/2 \rfloor$. This completes the proof. □

Definition 2. *(Nazeer et al., 2016) $C_{(m,0)}P_n$ is generalized caterpillar obtained from P_n by attaching m vertices of degree one to each vertex of degree two of P_n.*

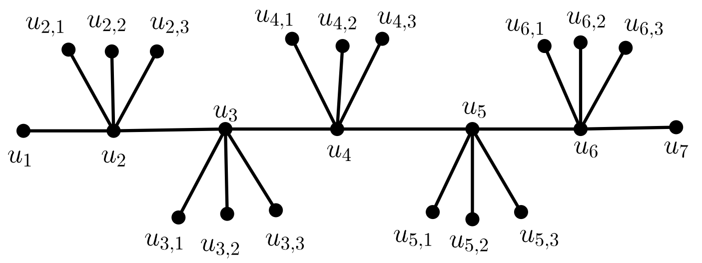

Figure 2: Caterpillar graph

Theorem 7. *Let G be a generalized caterpillar graph $C_{(m,0)}P_n$. Then, $IS(G) = (n-3)m+2$.*

Proof. The graph $C_{(m,0)}P_n$ is obtained from P_n by attaching the terminal vertex $u_{i,j}$ for $1 \leq j \leq m$ to the vertex u_i for each $2 \leq i \leq n-1$. Hence, we have three cases depending on the different vertex types of G.

Case 1. Let I be an $IS-set$ containing the vertex u of the graph G. If u is one of the leaf vertices of the graph G, then I consists only of leaves, since the leaf vertices are not adjacent to each other and are adjacent to vertex with degree 2 of the path. Hence, $IS(u) = (n-2)m+2$.

Case 2. Let I be an $IS-set$ containing the vertex u of the graph G. Let u be one of the support vertices of the graph P_n. Since $N_G(u_2) = \{1,\, u_{2,j}\, for\, j \in \{1,\, 2,\, \ldots, m\},\, u_3\}$ and $N_G(u_{n-1}) = \left\{n-2,\, u_{(n-1),j}\, for\, j \in \{1,\, 2,\, \ldots, m\},\, u_n\right\}$, then it can easily be seen that set I contains all the leaf vertices of the graph G except

$$\{u_1\,(or\;u_n)\,, u_{i,j} for\; i = 2 or\; n-1, j \in \{1, 2, ...m\}\,.$$

Hence, $IS(u) = (n-3)m+2$.

Case 3. Let I be an $IS-set$ containing the vertex u of the graph G. Let u be one of the vertices of the graph P_n except $\{u_1,\, u_2, u_{n-1}, u_n\}$. Since for $t \in \{3, 4..., n-2\}$ $N_G(u_t) = \{u_{t-1},\, u_{t,j}\, for\, j \in \{1,\, 2,\, \ldots, m\},\, u_t\}$, then it can easily be seen that set I contains all the leaf vertices of the graph except $\{u_{t,j} \in N_G(u_t)\}$. Hence, $IS(u) = (n-3)m+2$.

By *Cases 1, 2*, and *3*, we have $IS(G) = \min\{(n-2)m+2, (n-3)m+2\} = (n-3)m+2$. This completes the proof. □

Definition 3. *(Grossman et al., 1979) The double star $S_{a,b}$, where $a, b \geq 0$, is the graph consisting of the union of two stars $S_{1,a}$ and $S_{1,b}$ together with an edge joining their centers.*

Theorem 8. *Let G be a double star graph $S_{a,b}$. Then, $IS(G) = \min\{a, b\} + 1$.*

Proof. Let $V(S_{a,b}) = V(S_{1,a}) \cup V(S_{1,b})$. Let the center vertices of $S_{1,a}$ and $S_{1,b}$ be u_1 and u_2, respectively. Hence, we have three cases depending on the different vertex types of G.

Case 1. Let I be an $IS-set$ containing the vertex u_1 of the graph G. Since the vertex u_1 is adjacent to $(V(S_{1,a}) - \{u_1\})$ and the vertex u_2, $I = \{u_1\} \cup \{b_i \in V(S_{1,b}) :\; for\; i \in \{1, 2, \ldots, b\}\}$. This implies that $IS(u_1) = 1 + b$.

Case 2. Let I be an $IS-set$ containing the vertex u_2 of the graph G. Since the vertex u_2 is adjacent to $(V(S_{1,b}) - \{u_2\})$ and the vertex u_1, $I = \{u_2\} \cup \{a_j \in V(S_{1,m}) :\; for\; j \in \{1, 2, \ldots, a\}\}$. This implies that $IS(u_2) = 1 + a$.

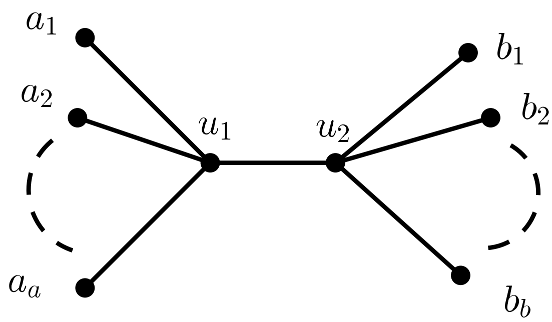

Figure 3: Double star graph

Case 3. Let I be an $IS-set$ containing the vertex $u \in (V(G) - \{u_1, u_2\})$ in the graph G. Since all the vertices in the set $(V(G) - \{u_1, u_2\})$ are pendant vertices, $I = \{a_1, \ldots, a_a, b_1, \ldots, b_b\}$. This implies that $IS(u) = a + b$.

By *Cases 1,2,* and *3*, we have $IS(G) = \min\{a+1, b+1, a+b\} = \min\{a, b\} + 1$. This completes the proof. □

Definition 4. *(Cormen et al., 1990) The binomial tree B_n is an ordered tree derived recursively. The binomial tree B_0 consists of a single vertex. The binomial tree B_n consists of two binomial trees B_{n-1} that are linked together: the root of one is the leftmost child of the root of the other.*

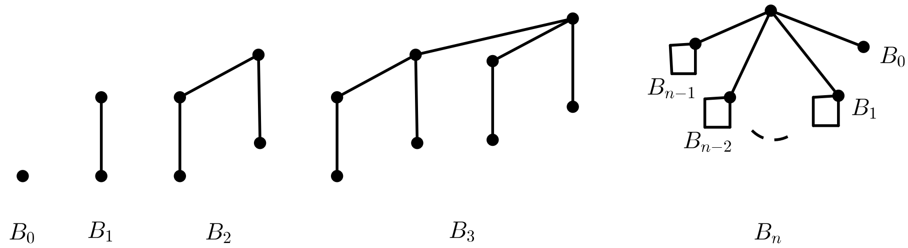

Figure 4: Binomial tree

Theorem 9. *Let G be a binomial tree B_n. Then, for $n \geq 1$, $IS(G) = 2^{n-1}$.*

Proof. B_n has 2^n vertices. The vertices of the graph consist of two sets, pendant vertices and support vertices. Let $S(G)$ and $P(G)$ be set the set of support vertices and pendant vertices, respectively. Hence, we have two cases depending on the different vertex types of G.

Case 1. Let I be an $IS-set$ containing a vertex $v \in P(G)$ in the graph G. Since one leaf vertex is not adjacent to all other leaf vertices, the maximum independent set containing the vertex v consists of all leaf vertices. Hence, $IS(v) = 2^{n-1}$.

Case 2. Let I be an $IS - set$ containing a vertex $v \in S(G)$ in the graph G. Since one of the neighbors of the vertex v is leaf vertex, $I = \{v\} \cup (P(G) - N_G(v))$. Hence, $IS(v) = 1 + 2^{n-1} - 1 = 2^{n-1}$.

By *Cases 1* and *2*, we have $IS(G) = 2^{n-1}$. This completes the proof. □

Definition 5. *(Cygan et al., 2011) For $a, b \geq 1$, $n \geq a+b+2$ by $DC(n, a, b)$ we denote a double comet, which is a tree composed of a path containing $n - a - b$ vertices with a pendant vertices attached to one of the ends of the path and b pendant vertices attached to the other end of the path. Thus, $DC(n, a, b)$ has n vertices and $a + b$ leaves.*

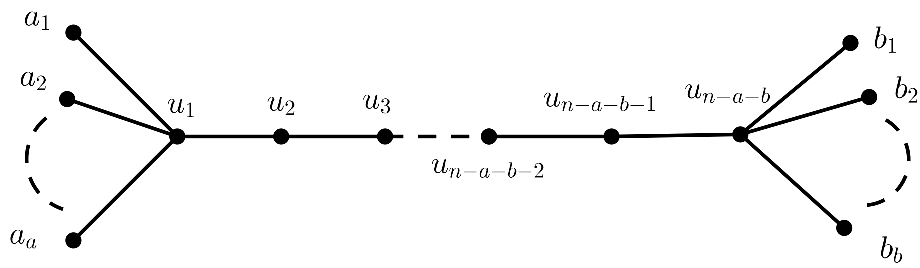

Figure 5: Double comet graph

Theorem 10. *Let G be a double comet $DC(n, a, b)$. Then,*

$$IS(G) = \lfloor (n - a - b)/2 \rfloor + \min\{a, b\}.$$

Proof. Let $V(G) = V(P_{n-a-b-2}) \cup V(S_{1,a}) \cup V(S_{1,b})$, where $V(P_{n-a-b-2}) = \{u_2, u_3, \ldots, u_{n-a-b-2}\}$, $V(S_{1,a}) = \{u_1, a_1, \ldots, a_a\}$ and $V(S_{1,b}) = \{u_{n-a-b}, b_1, \ldots, b_b\}$, u_1 and u_{n-a-b} are center vertices of $S_{1,a}$ and $S_{1,b}$, respectively. Hence, we have four cases depending on the different vertex types of G.

Case 1. Let I be an $IS - set$ containing a vertex $u \in V(P_{n-a-b-2})$ in the graph G. The set I is formed, as is the Theorem 5(a). Furthermore, the set I contains the all pendant vertices in the graph G. Hence, $IS(u) = \lfloor (n - a - b)/2 \rfloor + a + b$.

Case 2. Let I be an $IS - set$ containing a pendant vertex $u \in V(S_{1,a})$ or $u \in V(S_{1,b})$ in the graph G. In this case, the set I is equivalent to the set in *Case 1*. Hence, we have $IS(u) = \lfloor (n - a - b)/2 \rfloor + a + b$.

Case 3. Let I be an $IS - set$ containing the center vertex u_1 of $S_{1,a}$ in the graph G. Since the vertex u_1 is adjacent to all vertices a_i for $i \in \{1, 2, \ldots, a\}$, the set I does not include the vertices in the set $(V(S_{1,a}) - \{u_1\})$. It is easily seen that the set I contains all the leaf vertices of the graph $S_{1,b}$ and $\lfloor (n - a - b - 2)/2 \rfloor$ vertices in the graph $P_{n-a-b-2}$ due to Theorem 5(a). Then, we get $IS(u_1) = 1 + \lfloor (n - a - b - 2)/2 \rfloor + b = \lfloor (n - a - b)/2 \rfloor + b$.

Case 4. Let I be an $IS - set$ containing the center vertex u_2 of $S_{1,b}$ in the graph G. Since the vertex u_2 is adjacent to all vertices b_j for $j \in \{1, 2, \ldots, b\}$, the set I does not include the vertices in the set $(V(S_{1,b}) - \{u_2\})$. It is easily seen that the set I contains all the leaf vertices of the graph $S_{1,a}$ and $\lfloor (n - a - b - 2)/2 \rfloor$ vertices in the graph $P_{n-a-b-2}$ due to Theorem 5(a). Then, we get $IS(u_2) = 1 + \lfloor (n - a - b - 2)/2 \rfloor + a = \lfloor (n - a - b)/2 \rfloor + a$.

By *Cases 1, 2, 3* and *4* we have $IS(G) = \lfloor (n - a - b)/2 \rfloor + \min\{a, b\}$. This completes the proof. □

4 Conclusion

Independence saturation in some tree graphs are discussed in this paper. It gives us an idea of how the parameter will work in large graph structures containing the tree graphs we are considering and what the results might be.

References

Araujo, F., Farinha, J., Domingues, P., Silaghi, G. C., & Kondo, D. (2011). A maximum independent set approach for collusion detection in voting pools. *Journal of Parallel and Distributed Computing*, *71*(10), 1356 - 1366. Retrieved from `http://www.sciencedirect.com/science/article/pii/S0743731511001316` doi: https://doi.org/10.1016/j.jpdc.2011.06.004

Arumugam, S., & Subramanian, M. (2007). Independence saturation and extended domination chain in graphs. *AKCE Int. J. Graphs Comb.*, *4*(2), 171–181.

Berberler, Z. N., & Berberler, M. E. (2017). Independence saturation in complementary product types of graphs. *Celal Bayar University Journal of Science*, *13*, 325 - 331. doi: 10.18466/cbayarfbe.319783

Berberler, Z. N., & Berberler, M. E. (2018). Independent saturated graphs. *TWMS J. Appl. Eng. Math.*, *8*(1), 44–50.

Buckley, F., & Harary, F. (1990). *Distance in graphs.* Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA.

Butenko, S., Pardalos, P., Sergienko, I., Shylo, V., & Stetsyuk, P. (2002). Finding maximum independent sets in graphs arising from coding theory. In *Proceedings of the 2002 acm symposium on applied computing* (pp. 542–546). New York, NY, USA: ACM. Retrieved from `http://doi.acm.org/10.1145/508791.508897` doi: 10.1145/508791.508897

Chartrand, G., & Lesniak, L. (2005). *Graphs & digraphs* (4th ed.) [Book; Book/Illustrated]. Boca Raton : Chapman & Hall/CRC. (Formerly CIP)

Cormen, T. H., Leiserson, C. E., & Rivest, R. L. (1990). *Introduction to algorithms.* MIT Press, Cambridge, MA; McGraw-Hill Book Co., New York.

Cygan, M., Pilipczuk, M., & Škrekovski, R. (2011). Relation between Randić index and average distance of trees. *MATCH Commun. Math. Comput. Chem.*, *66*(2), 605–612.

Grossman, J. W., Harary, F., & Klawe, M. (1979). Generalized Ramsey theory for graphs. X. Double stars. *Discrete Math.*, *28*(3), 247–254. Retrieved from `https://doi.org/10.1016/0012-365X(79)90132-8` doi: 10.1016/0012-365X(79)90132-8

Joo, C., Lin, X., Ryu, J., & Shroff, N. B. (2016, may). Distributed greedy approximation to maximum weighted independent set for scheduling with fading channels. *IEEE/ACM Transactions on Networking*, *24*(03), 1476-1488. doi: 10.1109/TNET.2015.2417861

Karp, R. M. (1972). Reducibility among combinatorial problems. In *Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972)* (pp. 85–103).

Liu, Y., Lu, J., Yang, H., Xiao, X., & Wei, Z. (2015). Towards maximum independent sets on massive graphs. *Proceedings of the VLDB Endowment*, *8*(13), 2122–2133.

Nazeer, S., Khan, M. S., Kousar, I., & Nazeer, W. (2016). Radio labeling and radio number for generalized caterpillar graphs. *J. Appl. Math. Inform.*, *34*(5-6), 451–465. Retrieved from `https://doi.org/10.14317/jami.2016.451` doi: 10.14317/jami.2016.451

Pardalos, P. M., & Xue, J. (1994). The maximum clique problem. *J. Global Optim.*, *4*(3), 301–328. Retrieved from `https://doi.org/10.1007/BF01098364` doi: 10.1007/BF01098364